XVI. *Two general propositions in the method of differences. By* Thomas Knight, *Esq. Communicated by* Taylor Combe, *Esq. Sec. R. S.*

Read February 27, 1817.

1. Though so many ingenious writers have demonstrated, and, in various respects, extended the celebrated formulas of La Grange, for $\Delta^n \varphi(x)$, $\Sigma^n \varphi(x)$ no one appears to have entertained the idea, that these, and the more general cases, in which the quantities under the functional sign have their differences variable, might be included in one simple form.

Mr. Prony* is, I believe, the only mathematician who has given a form of any regularity to $\Delta^n \varphi(x)$, when the difference of x is variable; but he does not seem to have been aware of the capability of the method he was employing; and instead of embracing, as he might have done, all cases in one simple expression, he has proposed a formula which has neither any particular elegance in itself, nor any apparent relation to that which, in the simpler case, had been given by La Grange.

I suppose the truth of the differential equations

$$\Delta^n \varphi = \varphi_n - \frac{n}{1}\varphi_{n-1} + \frac{n(n-1)}{1.2}\varphi_{n-2} - \frac{n(n-1)\,(n-2)}{1.2.3}\varphi_{n-3} +, \quad (1)$$

$$\Sigma\, \varphi = \varphi_{-n} + \frac{n}{1}\varphi_{-(n+1)} + \frac{n(n+1)}{1.2}\varphi_{-(n+2)} + \frac{n(n+1)\,(n+2)}{1.2.3}\varphi_{-(n+3)} +, \ldots (2):$$

where φ is any variable function whatever.

* Lacroix "Calc. des Diff." p. 25.

PROP. I.

2. *To find the* n^{th} *difference of a function of any number of variable quantities,* $\Delta^n\,\varphi\,(x, y, z, \&c.)$, *when the differences of* x, y, z, &c. *are any how variable.*

We will begin with a function of two variables;

Let $x_1 - x = u_1,\ x_2 - x = u_2,\ x_3 - x = u_3, \ldots\ldots x_n - x = u_n$;
$y_1 - y = w_1,\ y_2 - y = w_2,\ y_3 - y = w_3, \ldots\ldots y_n - y = w_n$; } these values are simultaneous.

Let $\left(\frac{d^n\varphi(x,y)}{dx^n}\right) + \left(\frac{d^n\varphi(x,y)}{dx^{n-1}dy}\right) + \left(\frac{d^n\varphi(x,y)}{dx^{n-2}dy^2}\right) + \ldots\ldots + \left(\frac{d^n\varphi(x,y)}{dy^n}\right)$ be represented by $\Sigma\left(\frac{d^n\varphi(x,y)}{dx^{n-m}dy^m}\right)$; the sign Σ expressing *here* the sum of all the different values that will arise to the function within the brackets, by giving successively to m the values 0, 1, 2, 3, ... n

Lastly, let the symbol ⊠ represent what may be called elective multiplication; thus $\Sigma\left(\frac{d^n\varphi(x,y)}{dx^{n-m}dy^m}\right) \boxtimes (u+w)^n$ will denote that each value of $\left(\frac{d^n\varphi(x,y)}{dx^{n-m}dy^m}\right)$ is to be multiplied by the corresponding term of the expanded binomial $(u+w)^n$; viz. $\left(\frac{d^n\varphi(x,y)}{dx^n}\right)$ by u^n, $\left(\frac{d^n\varphi(x,y)}{dx^{n-1}dy}\right)$ by $nu^{n-1}w$, $\left(\frac{d^n\varphi(x,y)}{dx^{n-2}dy^2}\right)$ by $\frac{n(n-1)}{1.2}u^{n-2}w^2$, and so on Then

$$\varphi(x+u_1, y+w_1) = \varphi(x,y) + \Sigma\left(\frac{d\varphi(x,y)}{dx^{1-m}dy^m}\right) \boxtimes (u_1+w_1) + \frac{1}{2}\,.$$
$$\Sigma\left(\frac{d^2\varphi(x,y)}{dx^{2-m}dy^m}\right) \boxtimes (u_1+w_1)^2 + \frac{1}{2.3}\,.\,\Sigma\,.\left(\frac{d^3\varphi(x,y)}{dx^{3-m}dy^m}\right) \boxtimes (u_1+w_1)^3 +$$

$$\varphi(x+u_2, y+w_2)=\varphi(x,y)+\Sigma\left(\frac{d\varphi(x,y)}{dx^{1-m}dy^m}\right)\boxtimes(u_2+w_2)+\frac{1}{2}\cdot$$
$$\Sigma\left(\frac{d^2\varphi(x,y)}{dx^{2-m}dy^m}\right)\boxtimes(u_2+w_2)^2+\frac{1}{2.3}\cdot\Sigma\left(\frac{d^3\varphi(x,y)}{dx^{3-m}dy^m}\right)\boxtimes(u_2+w_2)^3+$$
$$\varphi(x+u_3, y+w_3)=\varphi(x,y)+\Sigma\left(\frac{d\varphi(x,y)}{dx^{1-m}dy^m}\right)\boxtimes(u_3+w_3)+\frac{1}{2}\cdot$$
$$\Sigma\left(\frac{d^2\varphi(x,y)}{dx^{2-m}dy^m}\right)\boxtimes(u_3+w_3)^2+\frac{1}{2.3}\cdot\Sigma\left(\frac{d^3\varphi(x,y)}{dx^{3-m}dy^m}\right)\boxtimes(u_3+w_3)^3+$$

&c. &c.

These values substituted in the equation $\Delta^n\varphi=\varphi_n-\frac{n}{1}\cdot\varphi_{n-1}+\frac{n(n-1)}{1.2}\cdot\varphi_{n-2}-\frac{n(n-1)(n-1)}{1.2.3}\cdot\varphi_{n-3}+$ give

$$\Delta^n\varphi(x,y)=\varphi(x,y)\left\{1-\frac{n}{1}+\frac{n(n-1)}{1.2}-\frac{n(n-1)(n-2)}{1.2.3}+\right\}$$
$$+\Sigma\left(\frac{d\varphi(x,y)}{dx^{1-m}dy^m}\right)\boxtimes\left\{(u_n+w_n)-\frac{n}{1}(u_{n-1}+w_{n-1})\right.$$
$$\left.+\frac{n(n-1)}{1.2}(u_{n-2}+w_{n-2})-\frac{n(n-1)(n-2)}{1.2.3}(u_{n-3}+w_{n-3})+\right\}$$
$$+\frac{1}{2}\Sigma\left(\frac{d^2\varphi(x,y}{dx^{2-m}dy^m}\right)\boxtimes\left\{(u_n+w_n)^2-\frac{n}{1}(u_{n-1}+w_{n-1})^2\right.$$
$$\left.+\frac{n(n-1)}{1.2}(u_{n-2}+w_{n-2})^2-\frac{n(n-1)(n-2)}{1.2.3}(u_{n-3}+w_{n-3})^2+\right\}$$
$$+\cdot\frac{1}{2.3}\Sigma\left(\frac{d^3\varphi(x,y)}{dx^{3-m}dy^m}\right)\boxtimes\left\{(u_n+w_n)^3-\frac{n}{1}(u_{n-1}+w_{n-1})^3\right.$$
$$\left.+\frac{n(n-1)}{1.2}(u_{n-2}+w_{n-2})^3-\frac{n(n-1)(n-2)}{1.2.3}(u_{n-3}+w_{n-3})^3+\right\}$$

&c. &c.

3. Now, by the general formula for the n^{th} difference of a variable quantity, (1),

$\Delta^n(x+y)^m=(x+y+u_n+w_n)^m-\frac{n}{1}(x+y+u_{n-1}+w_{n-1})^m+\frac{n(n-1)}{1.2}(x+y+u_{n-2}+w_{n-2})^m-$ whence, making x and y vanish,

$$\Delta^n(o+o')^m = (u_n + w_n)^m - \frac{n}{1}(u_{n-1} + w_{n-1})^m + \frac{n(n-1)}{1.2}(u_{n-2} + w_{n-2})^m - ;$$

the successive values of o being $u_1, u_2, u_3, \ldots\ldots u_n$

those of o' being $w_1, w_2, w_3, \ldots\ldots w_n$

observing therefore that $1 - \frac{n}{1} + \frac{n(n-1)}{1.2} - \frac{n(n-1)(n-2)}{1.2.3} + = \Delta^n(o+o')^0$, and putting $\Sigma\left(\frac{d^0\varphi(x,y)}{dx^{0-m}dy^m}\right)$ for $\varphi(x,y)$ our equation takes this form

$$\Delta^n\varphi(x,y) = \Sigma\left(\frac{d^0\varphi(x,y)}{dx^{0-m}dy^m}\right) \boxtimes \Delta^n(o+o')^0 + \Sigma\left(\frac{d^1\varphi(x,y)}{dx^{1-m}dy^m}\right) \boxtimes \Delta^n(o+o')^1 + \frac{1}{2}\Sigma\left(\frac{d^2\varphi(x,y)}{dx^{2-m}dy^m}\right) \boxtimes \Delta^n(o+o')^2 + \frac{1}{2.3}\,.\,\Sigma\left(\frac{d^3\varphi(x,y)}{dx^{3-m}dy^m}\right) \boxtimes \Delta^n(o+o')^3 + .$$

What has been done with respect to a function of two variables, the analyst will immediately see how to extend to a function containing any number; we may therefore without entering into any farther particulars, give the following

General Rule.

Let the successive values

of o be $u_1, u_2, u_3, \ldots\ldots u_n$

of o' be $w_1, w_2, w_3, \ldots\ldots w_n$

of o'' be $v_1, v_2, v_3, \ldots\ldots v_n$ *

... &c. .... ,*then will*

$$\Delta^n\varphi(x, y, z, \&c.) = \Delta^n e^{o+o'+o''+\&c.} \qquad (3)$$

provided that, after the expansion, we multiply every where a term of the form $A \times u^a_{n-r} \times w^b_{n-r} \times v^c_{n-r} \times$ &c. *by*

* Supposing $z_n - z = v_n$, &c. &c.

$$\left(\frac{d^{a+b+c+\&c.}\varphi(x,y,z,\&c.)}{dx^a.dy^b.dz^c.\&c.}\right).$$

4. The expression of LA GRANGE is a particular case of eq. (3), to perceive which we must observe that

$$\Delta^n e^{x+y+z+\&c.} = e^{x+y+z+\&c.+u_n+w_n+v_n+\&c.} - \frac{n}{1}.e^{x+y+z+\&c.+u_{n-1}+w_{n-1}+v_{n-1}+\&c.} + \frac{n(n-1)}{1.2}e^{x+y+z+\&c.+u_{n-2}+w_{n-2}+v_{n-2}+\&c.} + \&c. \text{ whence}$$

$$\Delta^n e^{o+o'+o''+\&c.} = e^{u_n+w_n+v_n+\&c.} - \frac{n}{1}e^{u_{n-1}+w_{n-1}+v_{n-1}} + \&c. + \frac{n(n-1)}{1.2}.e^{u_{n-2}+w_{n-2}+v_{n-2}+\&c.} - \&c.;$$

which, if x, y, z, &c. have constant differences, or if u^n, u^{n-1}, u_{n-2}, &c. w_n, w_{n-1}, w_{n-2}, &c. v_n, v_{n-1}, v_{n-1} &c., &c. are nu, $(n-1)u$, $(n-2)u$, &c., nw, $(n-1)w$, $(n-2)w$, &c., nv, $(n-1)v$, $(n-2)v$, &c. &c. becomes

$$\left\{e^{u+w+v+\&c.}-1\right\}^n.$$

The equation (3) may be presented under another form; for if we compare the values of

$$\Delta^n e^{x+y+z+\&c.} \text{ and } \Delta^n e^{o+o'+o''+\&c.} \text{ we see that}$$

$$\Delta^n e^{o+o'+o''+\&c.} = \frac{\Delta^n e^{x+y+z+\&c.}}{e^{x+y+z+\&c.}}, \text{ consequently}$$

$$\Delta^n\varphi(x,y,z,\&c.) = \frac{\Delta^n e^{x+y+z+\&c.}}{e^{x+y+z+\&c.}} \qquad (4)$$

where we must observe, with respect to the differential coefficients, the same rule as was given with eq. (3).

Prop. II.

5. *To find* $\Sigma^n \varphi(x, y, z, \&c.)$ *supposing that the differences of* x, y, z, &c. *are any how variable.*

We shall here make use of the same notation as we did in Prop. I, only let the preceding values

of x be $x-u_{-1}$, $x-u_{-2}$, $x-u_{-3}$, &c.
of y be $y-w_{-1}$, $y-w_{-2}$, $y-w_{-3}$, &c.
of z be $z-v_{-1}$, $z-v_{-2}$, $z-v_{-3}$, &c.
&c. &c.

It will be sufficient also to consider the case of two variable quantities, as was done in Prop. I.

First, we have, in general,

$$\varphi(x-u_{-r}, y-w_{-r}) = \varphi(x, y) - \Sigma\left(\frac{d\varphi(x, y)}{dx^{1-m}dy^m}\right) \boxtimes (u_{-r}+w_{-r}) + \\ + \frac{1}{2} \cdot \Sigma\left(\frac{d^2\varphi(x, y)}{dx^{2-m}dy^m}\right) \boxtimes (u_{-r}+w_{-r})^2 - \frac{1}{2.3} \cdot \Sigma\left(\frac{d^3\varphi(x, y)}{dx^{3-m}dy^m}\right) \\ \boxtimes (u_{-r}+w_{-r})^3 +$$

which expression being combined with (2), putting for the sake of symmetry $\varphi(x, y) = \Sigma\left(\frac{d^0\varphi(x, y)}{dx^{0-m}dy^m}\right)$, gives

$$\Sigma^n \varphi(x, y) = \Sigma\left(\frac{d^0\varphi(x, y)}{dx^{0-m}dy^m}\right) \boxtimes \left\{(u_{-n}+w_{-n})^0 + \frac{n}{1} \cdot (u_{-(n+1)} \right. \\ \left. + w_{-(u+1)})^0 + \frac{n(n+1)}{1.2}(u_{-(n+2)}+w_{-(n+2)})^0 + \&c. \right\} \\ - \Sigma\left(\frac{d^1\varphi(x, y)}{dx^{1-m}dy^m}\right) \boxtimes \left\{(u_{-n}+w_{-n})^1 + \frac{n}{1} \cdot (u_{-(n+1)} \right. \\ \left. + w_{-(n+1)})^1 + \frac{n(n+1)}{1.2}(u_{-(n+2)}+w_{-(n+2)})^1 + \&c. \right\}$$

$$+\frac{1}{2}\cdot\Sigma\left(\frac{d^2\varphi(x,y)}{dx^{2-m}dy^m}\right)\boxtimes\left\{(u_{-n}+w_{-n})^2+\frac{n}{1}\cdot(u_{-(n+1)}+w_{-(n+1)})^2 + \frac{n(n+1)}{1.2}(u_{-(n+2)}+w_{-(n+2)})^2+\&c.\right\}$$

$+$ &c.

But by equat. (2) $\Sigma^n(x+y)^m=(x+y-u_{-n}-w_{-n})^m+\frac{n}{1}\cdot(x+y-u_{-(n+1)}-w_{-(n+1)})^m+\frac{n(n+1)}{1.2}(x+y-u_{-(n+2)}-w_{-(n+2)})^m+$ whence

$$\Sigma^n(o+o')^m=\pm\left\{(u_{-n}+w_{-n})^m+\frac{n}{1}(u_{-(n+1)}+w_{-(n+1)})^m+\frac{n(n+1)}{1.2}\cdot(u_{-(n+2)}+w_{-(n+2)})^m+\right\}$$

The upper or lower sign having place accordingly as m is even or odd. So that our equation may be expressed thus,

$$\Sigma^n\varphi(x,y)=\Sigma\left(\frac{d^0\varphi(x,y)}{dx^{0-m}dy^m}\right)\boxtimes\Sigma^n(o+o')^0+\Sigma\left(\frac{d^1\varphi(x,y)}{dx^{1-m}dy^m}\right)\boxtimes\Sigma^n(o+o')^1+\frac{1}{2}\cdot\Sigma\left(\frac{d^2\varphi(x,y)}{dx^{2-m}dy^m}\right)\boxtimes\Sigma^n(o+o')^2+\frac{1}{2.3}\cdot\Sigma\left(\frac{d^3\varphi(x,y)}{dx^{3-m}dy^m}\right)\boxtimes\Sigma^n(o+o')^3+$$

and the analyst will, without any trouble, see the truth of the following rule for a function of any number of variables.

Let the preceding values

of o be $-u_{-1}, -u_{-2}, -u_{-3}, \ldots\ldots -u_{-n}$, &c.

of o' be $-w_{-1}, -w_{-2}, -w_{-3}, \ldots\ldots -w_{-n}$, &c.

of o'' be $-v_{-1}, -v_{-2}, -v_{-3}, \ldots\ldots -v_{-n}$, &c.

&c. &c.

then will

$$\Sigma^n\varphi(x,y,z,\&c.)=\Sigma^n e^{o+o'+o''+\&c.} \qquad (5)$$

provided that, after expansion, we multiply a term of the form $A\times u^a_{-(n+r)}\times w^b_{-(n+r)}\times v^c_{-(n+r)}\times\&c.$ *by*

$$\left(\frac{d^{a+b+c+\&c.}\varphi(x,y,z,\&c.)}{dx^a\times dy^b\times dz^c\times\&c.}\right)$$

We have by form (2),

$$\Sigma^n e^{x+y+\;+\&c.} = e^{x+y+z+\&c.-u_{-n}-w_{-n}-v_{-n}-\&c.} +$$
$$\frac{n}{1} \cdot e^{x+y+z+\&c.-u_{-(n+1)}-w_{-(n+1)}-v_{-(n+1)}-\&c.} +$$
$$\frac{n(n+1)}{1.2} \cdot e^{x+y+z+\&c.-u_{-(n+2)}-w_{-(n+2)}-v_{-(n+2)}-\&c.} +$$

&c. . . . . so that

$$\Sigma^n e^{o+o'+o''+\&c.} = e^{-u_{-n}-w_{-n}-v_{-n}-\&c.} +$$
$$\frac{n}{1} \cdot e^{-u_{-(n+1)}-w_{-(n+1)}-v_{-(n+1)}-\&c.} + \qquad (6)$$
$$\frac{n(n+1)}{1.2} \cdot e^{-u_{-(n+2)}-w_{-(n+2)}-v_{-(n+2)}-\&c.} + \&c.$$

By comparing these expressions, it appears that

$$\Sigma^n e^{o+o'+o''+\&c.} = \frac{\Sigma^n e^{x+y+z+\&c.}}{e^{x+y+z+\&c.}}$$ and consequently, that

$$\Sigma^n \phi(x, y, z, \&c.) = \frac{\Sigma^n . e^{x+y+z+\&c.}}{e^{x+y+z+\&c.}}$$, provided, &c.

It is scarcely necessary for me to observe, that the second member of the equation marked (6) becomes $\left\{ e^{u+w+v+\&c.}_{\;-1} \right\}^{-n}$ in the case of constant differences of $x, y, z,$ &c.; for u_{-n}, $u_{-(n+1)}$, &c. become in this case, nu, $(n+1)\,n$, &c., and the w, *s* and v, *s* undergo a similar change.

The results of the preceding propositions may be brought into a very small compass, viz.

Δ^{-n} *representing* Σ^n, *the* n^th^ *difference or the* n^th^ *integral of a function of any number of variable quantities, and varying in any possible manner, will be expressed by the equation* $\Delta^n \phi(x, y, z, \&c.) = \frac{\Delta^n e^{x+y+z+\&c.}}{e^{x+y+z+\&c.}}$; *provided that after expansion, we multiply* &c. &c. &c.

6. SCHOLIUM.

We may find, in many cases, very elegant and regular expressions for $\Delta^n\varphi(x)$, by supposing $\varphi(x+u)$ to be expanded differently from the form given by TAYLOR's theorem: as, for instance,

If $\varphi(x+u)=\psi(x)+X.\chi(u)+X'\chi'(u)+X''\chi''(u)+$, (7)
where X, X′, X″, &c. *represent any functions whatever of* x, and $\varphi, \psi, \chi, \chi'$ &c. *any functions of the quantities they stand before, then* Δx, or u, *being constant,*

$\Delta^n\varphi(x)=X.\Delta^n.\chi(o)+X'.\Delta^n.\chi'(o)+X''.\Delta^n.\chi''(o)+$&c.: for

$$\varphi(x+nu)=\psi(x)+X.\chi(nu)+X'.\chi'(nu)+$$

$$-\frac{n}{1}.\varphi(x+\overline{n-1}.u)=-\frac{n}{1}.\psi(x)-\frac{n}{1}X.\chi(\overline{n-1}.u)-\frac{n}{1}.X'.\chi'(\overline{n-1}.u)-$$

$$+\frac{n(n-1)}{1.2}\varphi(x+\overline{n-2}.u)=\frac{n(n-1)}{1.2}\psi(x)+\frac{n(n-1)}{1.2}.X.\chi(\overline{n-2}.u)+\frac{n(n-1)}{1.2}.X'.\chi'(\overline{n-2}).u)+$$

&c. &c.

and because $\Delta^n.\varphi=\varphi_n-\frac{n}{1}\varphi_{n-1}+\frac{n(n-2)}{1.2}\varphi_{n-2}-$, this being added give

$\Delta^n.\varphi(x)=X.\Delta^n.\chi(o)+X'.\Delta^n.\chi'(o)+X''.\Delta^n.\chi''(o)+$ (8)

If form (7) soon terminates, the expressions for the differences are very simple, as in

Ex. 1. Sin. $(x+u)=$ sin. x cos. $u+$ cos. x, sin. u, which being compared with (7) gives X $=$ sin. x, X′$=$ cos. x, X″, &c.$=o$; $\chi(o)=$ cos. o, $\chi'(o)=$ sin. o, $\chi''(o)$, &c.$=o$; whence

$$\Delta^n.\text{ sin. } x=\text{sin. } x.\ \Delta^n.\text{ cos. } o+\text{cos. } x.\ \Delta^n.\text{ sin. } o.$$

Ex. 2. Tang. $(x+u)=$tang. $x+$ sec.$^2 x\{$tang. $u+$tang. x.

$\text{tang.}^2 u + \text{tang.}^2 x.\ \text{tang.}^3 u + \}$, see DELAMBRE, Preface to BORDA, p. 48, whence, by our expression,

$$\Delta^n.\ \text{tang.}\ x = \text{sec.}^2 x \{ \Delta^n \text{tang.}\ o + \text{tang.}\ x\ .\ \Delta^n.\ \text{tang.}^2 o + \text{tang.}^2 x.\ \Delta^n.\ \text{tang.}^3 o + \}$$

Ex. 3. $\text{L. sin.}\ (x + u) = \text{L. sin.}\ x + \text{L. cos.}\ u + \text{M} \{ \text{Cot.}\ x.\ \text{tang.}\ u - \frac{1}{2} \text{cot.}^2 x.\ \text{tang.}^2 u + \frac{1}{3} \text{cot.}^3 x\ \text{tang.}^3 u + \}$ DELAMBRE, Preface to BORDA, p. 45; comparing with (7) and (8) we find $\Delta^n\ .\ \text{L}\ .\ \text{sin}\ .\ x = \Delta^n.\ \text{L}\ .\ \text{cos}\ .\ o +$

$$\text{M} \{ \text{Cot.} x \Delta^n. \text{tang.} o - \frac{1}{2} \text{cot}^2. x \Delta^n. \text{tang.}^2 o + \frac{1}{3} \text{cot.}^3 x. \Delta^n. \text{tang.}^3 o - \}$$

In like manner, because

$$\text{L}\ .\ \text{cos}\ .\ (x + u) = \text{L}\ .\ \text{cos.}\ x + \text{L}\ .\ \text{cos.}\ u - \text{M} \{ \text{tang.}\ x.\ \text{tang.}\ u + \frac{1}{2}\ \text{tang.}^2\ x.\ \text{tang.}^2 u + \frac{1}{3}\ \text{tang.}^3\ x.\ \text{tang.}^3 u + \}$$

$$\Delta^n.\ \text{L}\ .\ \text{cos.}\ x = \Delta^n.\ \text{L}\ .\ \text{cos.}\ o - \text{M} \{ \text{tang.}\ x\ \Delta^n.\ \text{tang.}\ o + \frac{1}{2}\ \text{tang.}^2\ x\ \Delta^n.\ \text{tang.}^2 o + \frac{1}{3}\ \text{tang.}^3 x\ \Delta^n\ \text{tang.}^3 o + \}$$

Then, because $\text{L}\ .\ \text{tang.} = \text{L}\ .\ \text{sin.} - \text{L}\ .\ \text{cos.}$, $\Delta^n.\ \text{L}\ .\ \text{tang.} = \Delta^n.\ \text{L}\ .\ \text{sin.} - \Delta^n\ .\ \text{L}\ .\ \text{cos.}$, therefore

$$\Delta^n.\ \text{L}\ .\ \text{tang.}\ x = \text{M} \{ (\text{Cot.}\ x + \text{tang.}\ x)\ \Delta^n.\ \text{tang.}\ o - \frac{1}{2}\ (\text{cot.}^2\ x - \text{tang.}^2\ x)\ \Delta^n.\ \text{tang.}^2 o + \frac{1}{3}\ (\text{cot.}^3\ x + \text{tang.}^3\ x) \Delta^n.\ \text{tang.}^3\ o - \}$$

If these forms were to be used for interpolation, we should have to calculate, before the commencement of a Table, $\Delta\ \text{L}\ .\ \text{cos.}\ o$, $\Delta^2.\ \text{L}\ .\ \text{cos.}\ o$, &c.; $\Delta\ .\ \text{tang.}\ o$, $\Delta^2\ .\ \text{tang.}\ o$, &c.; $\Delta\ .\ \text{tang.}^2\ o$, $\Delta^2.\ \text{tang.}^2\ o$, &c., &c. These latter quantities are to be multiplied by M, and will then serve for calculating the whole Table.

If three differences are sufficient, we have, making $u = 1'$,*

* It is the decimal division of the circle which is supposed here.

$\Delta \,.\, L \,.\, \cos.\, o = -, 0^{8}53579$, $\Delta^{2} \,.\, L \,.\, \cos.\, o = -, 0^{7}107158$, $\Delta^{3} \,.\, L \,.\, \cos.\, o =, o^{13}$;

$\Delta \,.\, \text{tang.}\, o =, 0^{3}1570796339$, $\Delta^{2} \,.\, \text{tang.}\, o =, 0''78$, $\Delta^{3}.\, \text{tang.}\, o =, 0''78$;

$\Delta \,.\, \text{tang.}^{2}\, o =, 0^{7}246740$, $\Delta^{2}.\, \text{tang.}^{2}\, o =, 0^{7}493480$, $\Delta^{3}.\, \text{tang.}^{2}\, o = o, 0^{13}$;

$\Delta \,.\, \text{tang.}^{3}\, o =, 0''39$, $\Delta.^{2}\, \text{tang.}^{3}\, o =, 0^{10}232$, $\Delta^{3}.\, \text{tang.}^{3}\, o =, 0^{10}232$.

Suppose, for a particular example, we want the first three differences of L. sin. 50°, we have

$$\Delta\, L \,.\, \sin.\, 50^{\circ} = -, 0^{8}53579 + M\left\{, 0^{3}1570796339 -, 0^{7}123370 +, 0''13\right\}$$

$$\Delta^{2} L \,.\, \sin.\, 50^{\circ} = -, 0^{7}107158 + M\left\{, 0''78 -, 0^{7}246740 +, 0''77\right\}$$

$$\Delta^{3}.\, L \,.\, \sin.\, 50^{\circ} = M\left\{, 0''78 +, 0''77\right\} \text{ or}$$

$\Delta\ L \,.\, \sin.\, 50^{\circ} =, 0000682081030$, $\Delta^{2} \,.\, L \,.\, \sin \,.\, 50^{\circ} = -, 00000002\,14249$,

$\Delta^{3}.\, L \,.\, \sin.\, 50^{\circ} =, 0000000000068$.